Three rearrangements and medians

Yohei Tsutsui *
(Shinshu University)

Dedicated to Professor Yasuo Komori-Furuya on his sixtieth birthday

Abstract

Relations among non-increasing (right or left) continuous rearrangements and medians are discussed, and then equivalences of (quasi-)norms of weak Lebesgue space are given. Most part of the statements in this note are not results by the author. These are already known and applied in several literatures.

1 Introduction

In this article, we assume that f is a measurable function, from $\mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, satisfying $|f(x)| < \infty$ a.e. This property is fulfilled if $f \in L^{p,\infty}$, $(p \in (0,\infty))$, for example. For a measurable subset $\Omega \subset \mathbb{R}^n$, $|\Omega|$ means the n-dimensional Lebesgue measure of Ω. Let $E \subset \mathbb{R}^n$ be a non-empty measurable subset with finite volume $|E|$, and $\alpha \in (0,1)$. The distribution function of f is denoted by

$$d_f(\lambda) := |\{x \in \mathbb{R}^n; |f(x)| > \lambda\}| \geq 0$$

for $\lambda \in [0,\infty)$. This is right continuous on $[0,\infty)$. We define $\inf \emptyset := \infty$. We use the fact that $\mathbb{R}^n$, with the Lebesgue measure, is a non-atomic space.

We discuss pointwise relations among the following non-increasing rearrangements:

$$\begin{aligned} R[f](t) &:= \inf\{\lambda > 0 : d_f(\lambda) \leq t\} = \inf\{\lambda \geq 0 : d_f(\lambda) \leq t\} \geq 0 \\ L_1[f](t) &:= \inf\{\lambda > 0 : d_f(\lambda) < t\} = \inf\{\lambda \geq 0 : d_f(\lambda) < t\} \geq 0 \\ L_2[f](t) &:= \sup_{|A|=t} \inf_{x \in A} |f(x)| \geq 0, \end{aligned}$$

for $t \in (0,\infty)$. I think that $R[f]$ is the most popular one, and it is well known that $R[f]$ is right continuous on $(0,\infty)$ and is equimeasurable with $|f|$. The author had firstly encountered others in papers by Lerner [6] and [7], see also [1] and [3]. The main purpose of this note is to give proofs of fundamental facts for these rearrangements and median. I should emphasize that most part of propositions in this note are already known, although the author could not find proofs in literatures.

In Section 2, we give a proof of the following

$$R[f](t) \leq L_1[f](t) = L_2[f](t) \quad \text{for } t \in (0,\infty)$$

*Department of Mathematical Sciences, Faculty of Science, Shinshu University, Asashi 3-1-1, Matsumoto, Nagano, 390-8621, Japan. e-mail: tsutsui@shinshu-u.ac.jp

After that, we show the left continuity of $L[f] = L_1[f] = L_2[f]$ on $(0, \infty)$ and the equimeasurability of f and $L[f]$. Owning to their continuities, the inequality $L[f](t) \le R[f](t)$ fails in general. One can see this by considering simple functions.

The local analogy involving medians

$$R[f\chi_E](\alpha|E|) \le m_{|f|}(1-\alpha, E) \le L_1[f\chi_E](\alpha|E|) = L_2[f\chi_E](\alpha|E|)$$

is considered in Section 3. The median $m_f(\alpha, E)$ of f over E with $\alpha \in (0,1)$ is a real number satisfying

$$|\{E : f < m_f(\alpha, E)\}| \le \alpha|E| \quad \& \quad |\{E : f > m_f(\alpha, E)\}| \le (1-\alpha)|E|. \tag{1}$$

Median is not unique. To avoid this inconvenience, it is useful to consider the maximal median:

$$M_f(\alpha, E) := \max\{m \in \mathbb{R} : |\{E : f < m\}| \le \alpha|E|\}|.$$

Medians can be regarded as an average of f on E in the sense of $L^{1,\infty}$, on the other hand $|E|^{-1}\int_E f dx$ is an average in the sense of L^1. We see that $M_f(\alpha, E)$ is well-defined and a median.

In Section 4, we show that:

$$\|f\|^R_{L^{p,\infty}} = \|f\|^L_{L^{p,\infty}} \quad \& \quad \|f\|^R_{L^{p,\infty}(E)} = \|f\|^m_{L^{p,\infty}(E)} = \|f\|^L_{L^{p,\infty}(E)}$$

where

$$\begin{aligned}
\|f\|^R_{L^{p,\infty}} &:= \sup_{\lambda>0} \lambda d_f(\lambda)^{1/p} = \sup_{t>0} t^{1/p} R[f](t) \\
\|f\|^L_{L^{p,\infty}} &:= \sup_{t>0} t^{1/p} L[f](t) \\
\|f\|^R_{L^{p,\infty}(E)} &:= \|f\chi_E\|^R_{L^{p,\infty}} = \sup_{0<\alpha<1} (\alpha|E|)^{1/p} R[f\chi_E](\alpha|E|) \\
\|f\|^m_{L^{p,\infty}(E)} &:= \sup_{0<\alpha<1} (\alpha|E|)^{1/p} m_{|f|}(1-\alpha, E) \text{ and} \\
\|f\|^L_{L^{p,\infty}(E)} &:= \|f\chi_E\|^L_{L^{p,\infty}} = \sup_{0<\alpha<1} (\alpha|E|)^{1/p} L[f\chi_E](\alpha|E|).
\end{aligned}$$

These equalities are one of motivations of this note. In Section 5, we consider pointwise estimates for distribution functions of maximal operators defined by rearrangements and medians. Finally, we give a proof of a fundamental fact for non-atomic space, which is applied in this note.

I would appreciate it if you could give me some comments or point out mistakes in this note.

2 Three rearrangements

2.1 Inequalities among three rearrangements

In this subsection, we show the following.

Proposition 2.1. *For any $t \in (0, \infty)$,*

$$R[f](t) \le L_1[f](t) = L_2[f](t). \tag{2}$$

Remark 2.1. *The first inequality still holds at* $t = 0$*, because* $L_1[f](0) = \infty$*. On the other hand, the second one fails at* $t = 0$*. For example, if* $f \equiv 1$*, then* $L_1[f](0) = \infty$ *and* $L_2[f](0) = 1$*.*

Proof. The first inequality is obvious from definitions.

We shall show $L_1[f](t) \leq L_2[f](t)$ for $t \in (0, \infty)$. In the case $L_1[f](t) = \infty$, because $d_f(\lambda) \geq t$ for any $\lambda > 0$ and $(\mathbb{R}^n; dx)$ is a non-atomic space, there is a subset $A \subset \{|f| > \lambda\}$ such that $|A| = t$, which implies $L_2[f](t) \geq \lambda$. Thus, $L_2[f](t) = \infty$. In the case $L_1[f](t) < \infty$, it holds $d_f(L_1[f](t) - \varepsilon) \geq t$ for any $\varepsilon > 0$. Similarly as above, there exists a subset $B \subset \{|f| > L_1[f](t) - \varepsilon\}$ so that $|B| = t$, which yields

$$L_2[f](t) \geq \inf_{x \in B} |f(x)| \geq L_1[f](t) - \varepsilon.$$

Therefore, $L_1[f](t) \leq L_2[f](t)$.

Next, we shall prove the inequality in the opposite direction: $L_1[f](t) \geq L_2[f](t)$. We may assume that there is $\lambda \in (0, \infty)$ such that $d_f(\lambda) < t$, because if this fails, $L_1[f](t) = \inf \emptyset = \infty$. We observe that every $C \subset \mathbb{R}^n$ with $|C| = t$ have $x_C \in C$ satisfying $|f(x_C)| \leq \lambda$. In fact, $|f| > \lambda$ on C means $|C| < t$. Hence, $L_2[f](t) \leq \lambda$, and then $L_2[f](t) \leq L_1[f](t)$. □

2.2 Right/left continuity

It is well-known that $R[f]$ is a right continuous function on $(0, \infty)$.

Proposition 2.2. $L[f]$ *is left continuous on* $(0, \infty)$*.*

Proof. We show the continuity of $L_1[f]$.

Fix $t \in (0, \infty)$ and we show the left continuity at t. It is sufficient to consider the case $L_1[f](t) < \infty$. From the definition of $L_1[f]$, we see that $d_f(L_1[f](t) + \varepsilon) < t$ for any $\varepsilon > 0$. Thus, there is $\delta \in (0, t)$ so that

$$d_f(L_1[f](t) + \varepsilon) < t - \delta.$$

This means that $L_1[f](t - \delta) \leq L_1[f](t) + \varepsilon$. □

2.3 Equimeasurability

Equimeasurability of f and $R[f]$ is well-known. Here, we prove the same fact with $L[f]$.

Proposition 2.3. *L[f] is equimeasurable with* f*, that is*

$$d_f(\lambda) = d_{L[f]}(\lambda) \text{ for } \lambda \in [0, \infty).$$

Proof. We prove this for $L_2[f]$.

Fix $\lambda \in [0, \infty)$ and denote

$$\Omega_\lambda := \{t \in (0, \infty); L_2[f](t) > \lambda\}.$$

Ω_λ is one of $\emptyset$, $(0, T)$, $(0, T]$ or $(0, \infty)$ with some $T \in (0, \infty)$.

∘ Case: $\Omega_\lambda = \emptyset$. In this case, since $d_{L_2[f]}(\lambda) = 0$ it is enough to show $d_f(\lambda) = 0$. We assume that $d_f(\lambda) > 0$. Then, from the right continuity of the distribution function, there are $\tau \in (0, \infty)$ and $\delta \in (0, \infty)$ such that $d_f(\lambda + \delta) > \tau$. We can find a measurable subset $A \subset \{|f| > \lambda + \delta\}$ satisfying $|A| = \tau$. This fact implies $L_2[f](\tau) \geq \lambda + \varepsilon$ that is a contradiction.

∘ Case: $\Omega_\lambda = (0,T)$ or $(0,T]$. In this case $T = \sup \Omega_\lambda$. For any $\varepsilon \in (0, T^{-1})$, it holds $L_2[f](T-\varepsilon) > \lambda$. Hence, there exists $A_\varepsilon \subset \mathbb{R}^n$ so that $|A_\varepsilon| = T - \varepsilon$ and $\inf_{x\in A_\varepsilon} |f(x)| > \lambda$. Thus $d_f(\lambda) \geq |A_\varepsilon| = T-\varepsilon$, which means $d_f(\lambda) \geq T$. If we assume that $d_f(\lambda) > T$, then there exist $t_0 \in (T, d_f(\lambda))$ and $\delta > 0$ such that $d_f(\lambda+\delta) > t_0$. Since we can find $B_\delta \subset \{|f| > \lambda + \delta\}$ fulfilling $|B_\delta| = t_0$, one has

$$L_2[f](t_0) \geq \inf_{x\in B_\delta} |f(x)| \geq \lambda + \delta,$$

and then a contradiction $t_0 \leq T$ occurs. Therefore, $d_f(\lambda) = T = |\Omega_\lambda| = d_{L_2[f]}(\lambda)$.

∘ Case: $\Omega_\lambda = (0,\infty)$. We shall prove $d_f(\lambda) \geq t$ for any $t \in (0,\infty)$. Because $L_2[f](t) > \lambda$ for $t \in (0,\infty)$, we have measurable subsets $\{A_t\}_{t\in(0,\infty)}$ enjoying $|A_t| = t$ and $\inf_{x\in A_t} |f(x)| > \lambda$, which yields $d_f(\lambda) \geq |A_t| = t$. □

3 Median

Median was firstly introduced by Carleson in [2]. All median satisfy also

$$|\{E : f < m_f(\alpha, E)\}| \leq \alpha|E| \quad \& \quad |\{E : f < m_f(\alpha, E)\}| \leq (1-\alpha)|E|.$$

Lemma 3.1. *If* $0 < |E| < \infty$, $m_{|f|}(\alpha, E) \geq 0$.

Proof. If $m_{|f|}(\alpha, E) < 0$, then

$$|E| = \left|\left\{E; |f| > m_{|f|}(\alpha, E)\right\}\right| \leq (1-\alpha)|E| < |E|.$$

□

The following convergence was proved by Fujii [4] in the case $\alpha = 1/2$, and Poelhuis and Torchinsky [8] in other cases.

Lemma 3.2.

$$\lim_{\substack{x\in Q \\ Q\searrow\{x\}}} m_f(\alpha, Q) = f(x) \ a.e.$$

Remark 3.1. *This should be compared with the Lebesgue differential theorem. Remark that this lemma does not need the integrability of* f.

We refer [8] for other properties of medians.

3.1 Maximal median

Before discussion on the relation among rearrangements and median, we clear that the maximal median $M_f(\alpha, E)$ is well-defined.

Proposition 3.1. *Let* $0 < |E| < \infty$. *For* $A := \{m \in \mathbb{R} : |\{E : f < m\}| \leq \alpha|E|\}$,

$$M_f(\alpha, E) := \max A < \infty$$

and $M_f(\alpha, E)$ *is a median of* f *over* E *with* α.

Proof. ∘ *Step 1: $A \neq \emptyset$.*
If $A = \emptyset$, then $|\{E : f < m\}| > \alpha|E|$ for any $m \in \mathbb{R}$. Hence, from $|f(x)| < \infty$ a.e., we have a contradiction:

$$0 = \left|\bigcap_{\ell\in\mathbb{N}}\{E : f < -\ell\}\right| = \lim_{\ell\to\infty} |\{E : f < -\ell\}| \geq \alpha|E|.$$

∘ *Step 2: $a := \sup A < \infty$.*
We assume that $|\{E : f < m\}| \leq \alpha|E|$ for any $m \in \mathbb{R}$. Therefore, the following conflict occurs

$$|E| = |\{E : f < \infty\}| = \lim_{\ell\to\infty} |\{E : f < \ell\}| \leq \alpha|E|,$$

and then one finds $m_0 \in \mathbb{R}$ so that $|\{E : f < m_0\}| > \alpha|E|$. Thus, $\sup A \leq m_0 < \infty$. Hence, there exists $a = \sup A$ and $a < \infty$.

∘ *Step 3: $a \in A$, i.e. $a = \max A = M_f(\alpha, E)$.*
This can be seen as follows:

$$|\{E : f < a\}| = \lim_{\ell\to\infty} |\{E : f < a - 1/\ell\}| \leq \alpha|E|.$$

Next, we check that the maximal median is in fact a median. Obviously, from the definition, we have

$$|\{E : f < M_f(\alpha, E)\}| \leq \alpha|E| \quad \& \quad |\{E : f \geq M_f(\alpha, E) + 1/\ell\}| < (1 - \alpha)|E|, \ (\ell \in \mathbb{N}).$$

The second inequality yields

$$|\{E : f > M_f(\alpha, E)\}| = \lim_{\ell\to\infty} |\{E : f \geq M_f(\alpha, E) + 1/\ell\}| \leq (1 - \alpha)|E|.$$

Therefore, the maximal median $M_f(\alpha, E)$ is a median. □

3.2 Pointwise inequalities for three rearrangements and medians

Here, we prove a local counterpart of Proposition 2.1 involving medians. The first two inequalities were proved by Poelhuis and Torchinsky [8], and the last equality can be showed from the same argument as Proposition 2.1.

Proposition 3.2. *If $0 < |E| < \infty$, then for all of medians $m_f(\alpha, E)$ it holds true that*

$$R[f\chi_E](\alpha|E|) \leq m_{|f|}(1 - \alpha, E) \leq L_1[f\chi_E](\alpha|E|) = L_2[f\chi_E](\alpha|E|). \tag{3}$$

Remark 3.2. *The first two inequalities was proved by Poelhuis and Torchinsky [8]. In there, they stated that the third one is bounded by the first one. But, testing a simple function f we see that this fails.*

Proof. For simplicity, we write $R = R[f\chi_E](\alpha|E|)$, $L_1 = L_1[f\chi_E](\alpha|E|)$ and $L_2 = L_2[f\chi_E](\alpha|Q|)$. Recall that $0 \leq m_{|f|}(1 - \alpha, E) \leq M_{|f|}(1 - \alpha, E) < \infty$ from Lemma 3.1 and Proposition **??**.

∘ *Step 1: $R \leq m_{|f|}(1 - \alpha, E)$.*
This is deduced from the definition of median: $|\{E; |f| > m_{|f|}(1 - \alpha, E)\}| \leq \alpha|E|$.

∘ *Step 2:* $m_{|f|}(1-\alpha, E) \le L_1$.
We may assume $L_1 \in [0,\infty)$. Fix $\varepsilon > 0$ and observe that $|\{E; |f| \le L_1+\varepsilon\}| > (1-\alpha)|E|$. If $L_1+\varepsilon < m_{|f|}(1-\alpha, E)$, then one has the contradiction

$$(1-\alpha)|E| < |\{E; |f| \le L_1+\varepsilon\}| \le (1-\alpha)|E|.$$

∘ *Step 3:* $L_1 = L_2$.
Because $(E; dx)$ is a non-atomic measure space, the same argument as that in the proof of Proposition 2.1 can work. □

4 Equivalence quasi-norms of $L^{p,\infty}$ with rearrangements and medians

In this section, using propositions in previous sections, we show equivalents of $L^{p,\infty}$-quasi-norms, which based on rearrangements and medians.

Proposition 4.1. *Let* $p \in (0,\infty)$.
(i)

$$\|f\|^R_{L^{p,\infty}} = \|f\|^L_{L^{p,\infty}}.$$

(ii)

$$\|f\|^R_{L^{p,\infty}(E)} = \|f\|^m_{L^{p,\infty}(E)} = \|f\|^L_{L^{p,\infty}(E)}.$$

Proof. (i): Since $R[f](t) \le L[f](t)$, $\|f\|^R_{L^{p,\infty}} \le \|f\|^L_{L^{p,\infty}}$. To show the opposite direction, we take $t \in (0,\infty)$ fulfilling $L[f](t) > 0$ and $\varepsilon \in (0, L[f](t))$. Therefore, we can see

$$\begin{aligned}\|f\|^R_{L^{p,\infty}} = \sup_{\lambda>0} \lambda d_f(\lambda)^{1/p} &\ge (L[f](t)-\varepsilon) d_f(L[f](t)-\varepsilon)^{1/p} \\ &\ge (L[f](t)-\varepsilon) t^{1/p}.\end{aligned}$$

Taking the limit $\varepsilon \to 0$, we have $\sup_{\substack{t>0 \\ L[f](t)>0}} t^{1/p} L[f](t) \le \|f\|^R_{L^{p,\infty}}$. Hence, $\|f\|^L_{L^{p,\infty}} \le \|f\|^R_{L^{p,\infty}}$.

(ii): It is sufficient to show that $\|f\|^R_{L^{p,\infty}(E)} = \|f\|^L_{L^{p,\infty}(E)}$. Moreover, it is enough to prove, from Proposition 3.2,

$$\|f\|^L_{L^{p,\infty}(E)} \le \|f\|^R_{L^{p,\infty}(E)}.$$

This is done from the same argument above with $f = f\chi_E$. □

5 Distribution function estimates for maximal operators of rearrangements and medians

In this section, we consider the boundedness of the following maximal operators: for $x \in \Omega$

$$\begin{cases} m^R_{\alpha,\Omega}[f](x) := \sup_{\Omega \supset Q \ni x} R[f\chi_Q](\alpha|Q|), \\ m^m_{\alpha,\Omega}[f](x) := \sup_{\Omega \supset Q \ni x} m_{|f|}(1-\alpha, Q) \text{ and} \\ m^L_{\alpha,\Omega}[f](x) := \sup_{\Omega \supset Q \ni x} L[f\chi_Q](\alpha|Q|), \end{cases}$$

where Ω is a measurable subset of $\mathbb{R}^n$ and the supremums are taken over all of cubes $Q \subset \Omega$ including x. Here, 'cube' means a cube whose slides are parallel to axes. When $\Omega = \mathbb{R}^n$, we abbreviate Ω. Proposition 3.2 says that

$$m^R_{\alpha,\Omega}[f](x) \le m^m_{\alpha,\Omega}[f](x) \le m^L_{\alpha,\Omega}[f](x)$$

for all $x \in \Omega$.

Proposition 5.1. *For any $\lambda > 0$,*

$$|\{\Omega; m^L_{\alpha,\Omega}[f] > \lambda\}| \le \alpha^{-1}\|M\|_{L^1\to L^{1,\infty}}|\{\Omega; |f| > \lambda\}|.$$

Consequently, $m^R_{\alpha,\Omega}$, $m^m_{\alpha,\Omega}$ and $m^L_{\alpha,\Omega}$ are bounded operators on $L^{p,q}(\Omega)$ for all $p \in (0,\infty)$ and $q \in (0,\infty]$.

Proof. If $\lambda < m^L_{\alpha,\Omega}[f](x)$, then there exists a cube $Q \subset \Omega$ containing x so that $L[f\chi_Q](\alpha|Q|) > \lambda$. Hence, we can find $E_Q \subset Q$ fulfilling $|E_Q| = \alpha|Q|$ and $\inf_{x\in E_Q} |f(x)| > \lambda$. Since $|\{Q; |f| > \lambda\}| \ge |E_Q| = \alpha|Q|$, we see $M(\chi_{\{\Omega;|f|>\lambda\}})(x) \ge \alpha$, where M is the Hardy-Littlewood maximal operator. Therefore,

$$\begin{aligned}
|\{\Omega; m^L_{\alpha,\Omega}[f] > \lambda\}| &\le |\{M(\chi_{\{\Omega;|f|>\lambda\}}) \ge \alpha\}| \\
&= \lim_{\ell\to\infty} \left|\left\{M(\chi_{\{\Omega;|f|>\lambda\}}) > \alpha - 1/\ell\right\}\right| \\
&\le \lim_{\ell\to\infty} \frac{\|M\|_{L^1\to L^{1,\infty}}}{\alpha - 1/\ell}\|\chi_{\{\Omega;|f|>\lambda\}}\|_{L^1} \\
&= \alpha^{-1}\|M\|_{L^1\to L^{1,\infty}}|\{\Omega; |f| > \lambda\}|.
\end{aligned}$$

Consequently, we get $\|m^L_{\alpha,\Omega}[f]\|_{L^{1,\infty}(\Omega)} \le \alpha^{-1}\|M\|_{L^1\to L^{1,\infty}}\|f\|_{L^{1,\infty}(\Omega)}$. □

6 Appendix

Let $(X, |\cdot|)$ be a measure space with $|X| < \infty$. A measurable subset $E \subset X$ with $|E| > 0$ is an atom if the measure of any measurable subset $F \subset E$ is 0. If X has no atoms, then X is a non-atomic space.

Proposition 6.1. *Let $(X, |\cdot|)$ be a non-atomic measure space with $0 < |X| < \infty$. For any $\alpha \in (0, |X|]$, there exists $E \subset X$ so that $|E| = \alpha$.*

Proof. We may assume that $\alpha < |X|$.

∘ Claim: If $F \subset X$ and $0 < \beta \le |F|$, then there is $F_\beta \subset F$ such that $0 < |F_\beta| < \beta$.

Since F is not an atom, there is $A_1 \subset F$ satisfying $0 < |A_1| \le |F|$. Define

$$A_1^* := \begin{cases} A_1 & \text{if } |A_1| \le \frac{1}{2}|F| \\ F\backslash A_1 & \text{if } |A_1| > \frac{1}{2}|F|. \end{cases}$$

By the same argument, we can find $A_2^* \subset A_1^* \subset F$ fulfilling $0 < |A_2^*| \le \frac{1}{2}|A_1^*| \le \frac{1}{4}|F|$. Repeating this argument, this claim is verified.

From the claim, one has $E_1 \subset X$ so that $0 < |E_1| < \alpha$. Define

$$U_1 := \{B \subset X \backslash E_1; 0 < |B| < \alpha - |E_1|\} \text{ and } U_1' := \{B \in U_1; 1 \le |B|\}.$$

The claim ensures that $U_1 \neq \emptyset$. Let

$$E_2 \in \begin{cases} U_1' & \text{if } U_1' \neq \emptyset \\ U_1 & \text{if } U_1' = \emptyset. \end{cases}$$

Similarly, we define

$$U_2 := \{B \subset X \backslash (E_1 \cup E_2); 0 < |B| < \alpha - |E_1 \cup E_2|\} \text{ and } U_2' := \left\{B \in U_2; \frac{1}{2} \le |B|\right\},$$

and then take

$$E_3 \in \begin{cases} U_2' & \text{if } U_2' \neq \emptyset \\ U_2 & \text{if } U_2' = \emptyset. \end{cases}$$

Repeating this, we can get $\{E_m\}_{m=1}^{\infty} \subset X$ satisfying either

$$E_{m+1} \in U_m := \left\{B \subset X \backslash (E_1 \cup \cdots \cup E_m); 0 < |B| < \alpha - \sum_{j=1}^{m} |E_j|\right\} \neq \emptyset$$

or

$$E_{m+1} \in U_m' := \left\{B \in U_m; \frac{1}{m} \le |B|\right\}.$$

Moreover, the last case occurs if and only if $U_m' \neq \emptyset$. $E := \cup_{m=1}^{\infty} E_m$ enjoys $|E| \le \alpha$. In the case $|E| = \alpha$, the proof is completed. In the case $|E| < \alpha$, we have $|X \backslash E| \ge \alpha - |E| > 0$. Hence from the claim, there is $F \subset X \backslash E$ so that

$$0 < |F| < \alpha - |E| < \alpha - \sum_{j=1}^{m} |E_j|$$

for any $m \in \mathbb{N}$. Since there is $M \in \mathbb{N}$ so that $\frac{1}{M} \le |F|$, $F \in U_m'$ for any $m \ge M$. Therefore, if $m \ge M$, then $\frac{1}{m} \le |E_m|$, which implies $|E| = \infty$. This contradicts with $|E| < \alpha$. Thus, $|E| = \alpha$. □

7 A problem from Prof. Nakai

After the talk, Prof. Nakai asked me the following problem.

- In the case $R[f\chi_E](\alpha|E|) < L[f\chi_E](\alpha|E|)$, are all real numbers between them median of f over E with $1 - \alpha$?

What I can say for this, up to now, is that $R[f\chi_E](\alpha|E|)$ is a median. This can be seen as follows. From the right continuity of the distribution, it holds

$$|\{E; |f| > R[f\chi_E](\alpha|E|)\}| = d_{f\chi_E}(R[f\chi_E](\alpha|E|)) \le \alpha|E|.$$

On the other hand, from Proposition 3.2, we have

$$|\{E; |f| < R[f\chi_E](\alpha|E|)\}| \leq \left|\left\{E; |f| < m_{|f|}(1-\alpha, E)\right\}\right| \leq (1-\alpha)|E|.$$

These means that $R[f\chi_E](\alpha|E|)$ is a median of f over E with $1-\alpha$. Combining Proposition 3.2, we know that $R[f\chi_E](\alpha|E|)$ is the minimum of such medians. It is not hard to see that if a real number a is between distinct medians, then a is also a median. From these, we know that the set of all medians of f over E with fixed α is a closed interval. But I do not know when the maximal median $M_{|f|}(1-\alpha, E)$ coincides with $L[f\chi_E](\alpha|E|)$. Of course, when $R[f\chi_E](\alpha|E|) = L[f\chi_E](\alpha|E|)$, $L[f\chi_E](\alpha|E|)$ is also a median.

Acknowledgements

The author was partially supported by JSPS, though Grand-in-Aid-for Young Scientists (B) (No. 15K20919).

References

[1] C. Bennett and R. Sharply, *Interpolation of Operators*, Pure and Applied Mathematics, vol. 129 (1988).

[2] L. Carleson, *BMO- 10 years' development*, 18th Scandinavian Congress of Mathematicians (Aarhus, 1980), pp. 3–21, Progr. Math., **11**, Birkhäuser, Boston, Mass., 1981.

[3] K.M. Chong and P.A. Rice, Equimeasurable rearrangements of functions, Queen's Papers in Pure and Appl. Math., vol. 28, Queen's Univ., Kingston, Ontario, (1971).

[4] N. Fujii, *A condition for a two-weight norm inequality for singular integral operators*, Studia Math. **98** (1991), no. 3, 175–190.

[5] L. Grafakos, *Classical Fourier analysis*, Graduate Texts in Mathematics, **249**, Third ed., Springer, New York, (2014).

[6] A.K. Lerner, *Maximal functions with respect to differential bases measuring mean oscillation*, Anal. Math., **24**, 41-58, (1998).

[7] A.K. Lerner, *On weighted estimates of non-increasing rearrangements*, East J. Approx., **4**, 277-290, (1998).

[8] J. Poelhuis and A. Torchinsky, *Medians, continuity, and vanishing oscillation*, Studia Math. **213** (2012), no. 3, 227–242.

[9] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclid Spaces*, Princeton Univ. Press, Princeton, NJ, (1971).